New method to prove Pythagorean Theorem

Completing Squares

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Completing Squares

Definition: Completing a square is a technique to transform a quadratic polynomial of the form $px^2 + qx + r$ into the form $p(x+h)^2 + k$ for some values of h and k. Here p, q, r, h and k are constants numerals implying their values do not depend on x.

In past, many applications of completing squares were identified such as solving quadratic equations, graphing quadratic functions, evaluating integrals in calculus such as Gaussian integrals with a linear term in the exponent, factoring polynomials and finding Laplace transforms. In the foregoing discussion we will expand the application base of completing square and apply it to prove famous Pythagorean theorem.

One can easily convert any a quadratic polynomial $px^2 + qx + r$ into a monic polynomial, the coefficient p of x^2 term to 1. In this monic form the quadratic polynomial is changed to

$$x^2 + (q/p)\ x + r/q \qquad (1)$$

which we can rewrite as $x^2 + bx + c$ where $b = q/p$ and $c = r/q$.

Next we will illustrate completing square procedure by considering a numeric example. The general formula for computing square from elementary algebra is

$$(x + s)^2 = (x + s) \times (x + s) = x^2 + 2sx + s^2 \qquad (2)$$

In a perfect square the number s on left hand side, $s = ½ \times 2s$, coefficient of term for x on right hand side and the constant $c = s^2$. Therefore b in equation (1) is $b = 2s$ in (2).

Consider a monic quadratic polynomial $x^2 + 14x + 56$. This polynomial is not a perfect square because square root of 56 is not a whole number and is not equal to square of 7. However it is possible to express this polynomial as sum of perfect square and a constant term. i. e.

$x^2 + 14x + 49 + 7 = (x + 7)^2 + 7$, this segregation of original polynomial in two parts is called **completing the square**. Thus any general polynomial of second order $x^2 + bx + c$ (the highest power of x =2) can be transformed to $(x + b/2)^2 + k$, where k is a constant.

For non-monic polynomial $px^2 + qx + r$, we can rewrite in completed square form as

$$px^2 + qx + r = p\ (x - h)^2 + k \text{ where } h = -q/(2p) \text{ and } k = r – q^2/4p \qquad (3)$$

When p =1, monic case, $x^2 + bx + c = (x - h)^2 + k$, where $h = -b/2$ and $k = c – b^2/4$

Thus for any monic polynomial it is possible to form a square that has first two terms.
For equation (3), we can show graphically that left hand side indeed equals right hand side (see Figure 1). For LHS $px^2 + qx + r$. For RHS $(x + q/2p)^2 + r - q^2/4p$

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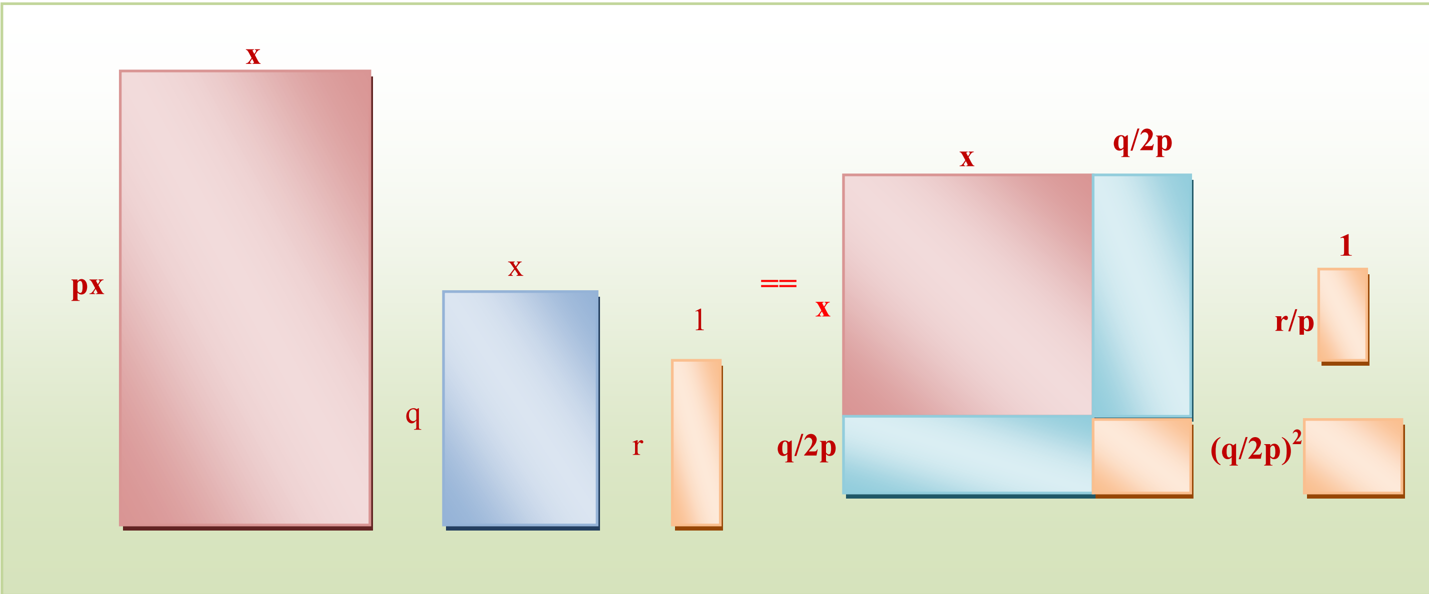

Figure 1. Area of any monic polynomial equals area of a perfect square plus a constant area.

Proof: Pythagorean Theorem

By Completing Squares

Next we will apply completing square method to specific case of Pythagorean theorem.

Pythagoras equation is a very special case of monic quadratic polynomial. The equation looks $a^2 + b^2 = c^2$. If we compare with standard monic quadratic polynomial $x^2 + bx + c$ we see resemblance. We find that $x = a$, $b = 0$ and $c = b^2 - c^2$. Therefore we can transform Pythagoras equation and complete the square. We shall use this method to prove that indeed $a^2 + b^2 = c^2$ for right angle triangle.

For right angled triangle ABC of figure 2 it is clear that area of square on hypotenuse c side is partitioned in four triangles whose area is ½ (ab) each plus a square whose area is $(a-b)^2$.

Therefore total area of square on hypotenuse is $c^2 = 4 \times ½(ab) + (a-b)^2$

For side a in the triangle ABC we can construct a square of side **a** and two rectangles of area ab
We can show by inspection of areas
$a^2 + b^2 = 2 \times ab + (a-b)^2$
Thus we proved Pythagorean theorem $\mathbf{a^2 + b^2 = c^2}$ **(1)**

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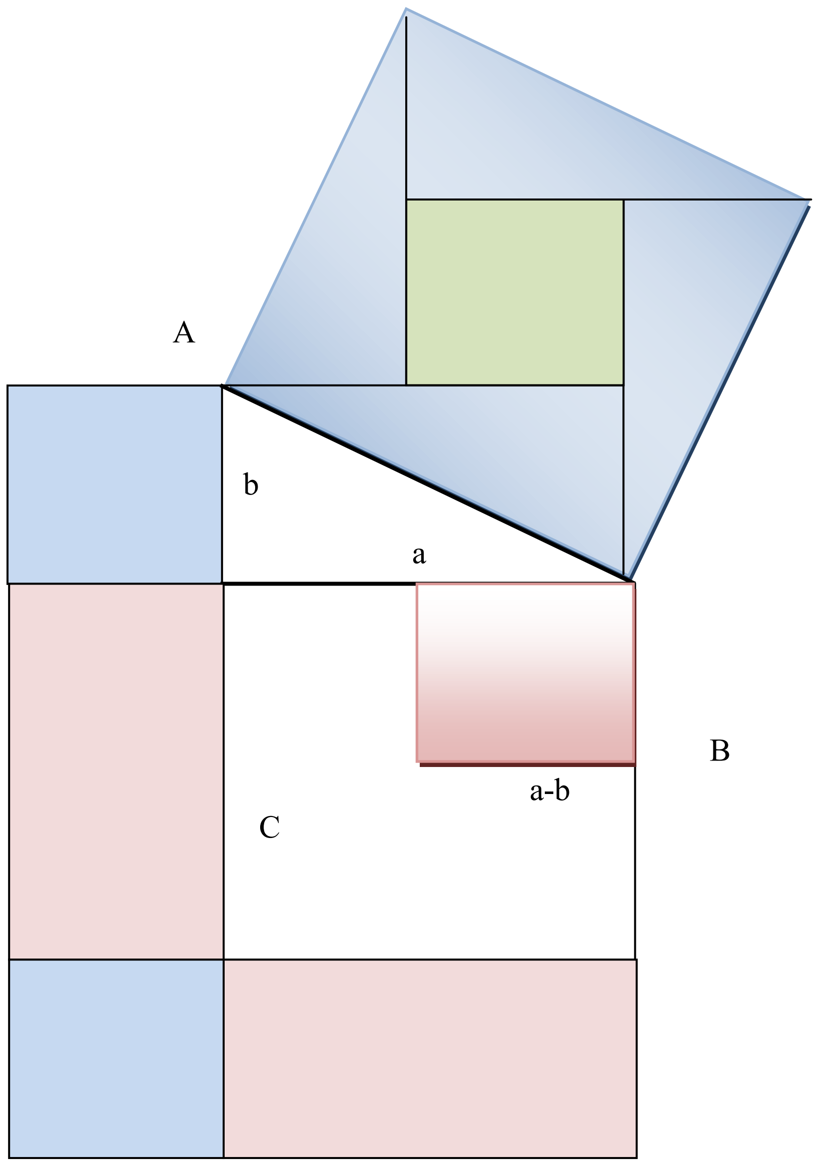

Figure 2.

Another Method: Geometric relationship among three sides of a triangle

Further we can prove the theorem by algebraic identity $(a + b)^2 - 2ab = a^2 + b^2$ as follows.

It is well known fact that sum of any two sides of any triangle is greater than the third side. In case of right angled triangle ABC of figure 2, length of side c is largest because angle C is 90°

This implies that sum of other two angles A and B is equal to 90°, each of the angles A and B are smaller than 90° and corresponding sides are smaller than side c. Mathematically

side $c < a + b$, hence $c = a + bx$ where $0 < x < 1$. Therefore $c^2 = (a + bx)^2$

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From figure 3 by adding areas of rectangles and square we get $a^2 + abx + bx(a+bx) = c^2$

We can simplify above expression and solve for x
Upon simplification we get $b^2x^2 + 2abx - b^2 = 0$ to solve for x

$$\mathbf{x = -2ab \pm (4a^2b^2 + 4b^2b^2)^{1/2}/2b^2 = [-a \pm (a^2+b^2)^{1/2}]/b > 0}$$

We can add 2ab term in both sides of equation (1) so that we get
$a^2 + b^2 - 2ab = c^2 - 2ab$ which is same as $(a - b)^2 = c^2 - 2ab$ alternately
$a^2 + b^2 + 2ab = c^2 + 2ab$ which is same as $(a + b)^2 = c^2 + 2ab$

This process of adding 2ab in equation (1) to complete $\mathbf{(a + b)^2}$ is known as completing square

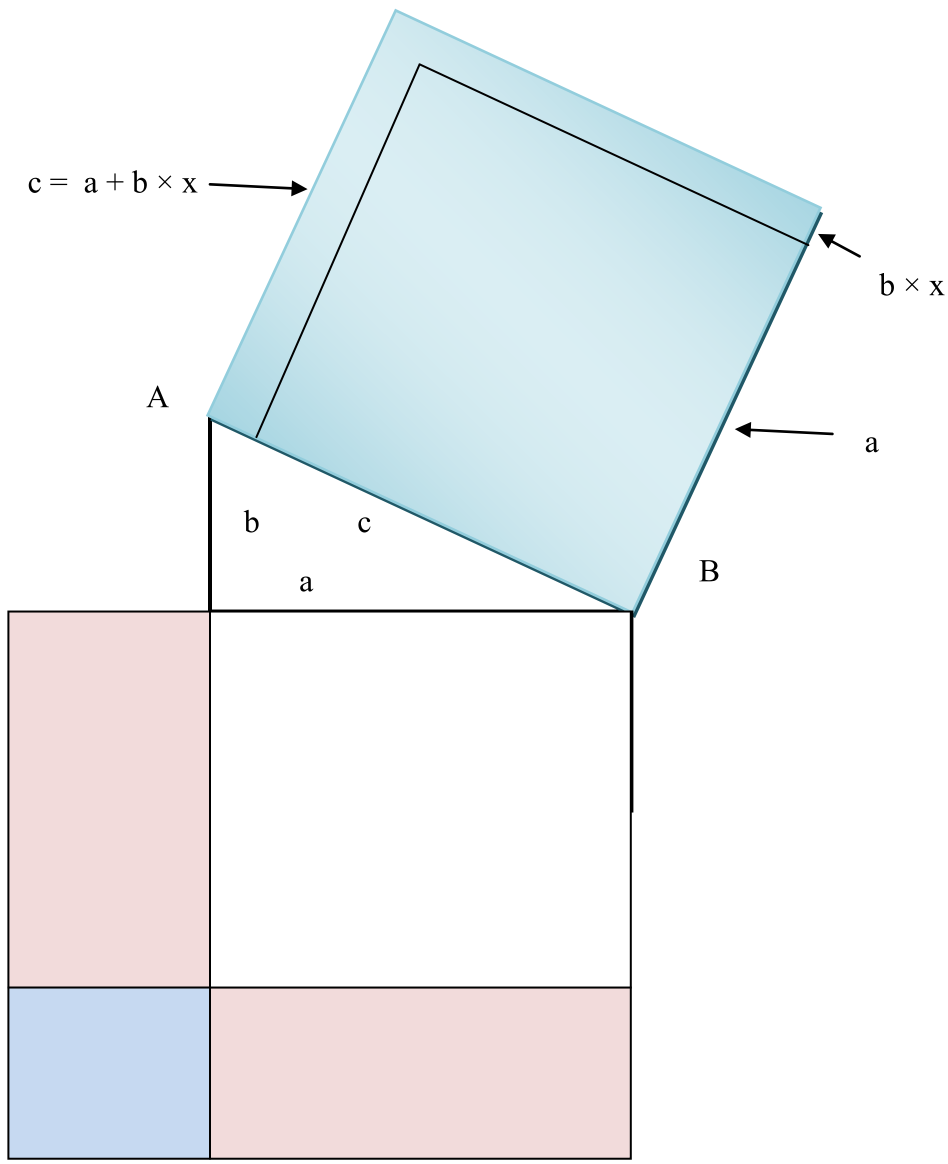

Figure 3.

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